$$V^{\cdot} = -h\,[(p_1 - \lambda\delta_1)^2 + (q_1 - \lambda\delta_2)^2 + (r_1 - \lambda\delta_3)^2]$$

The set $\{\mathbf{x} : \mathbf{y} = 0\}$ is invariant and the set $E \setminus \{\mathbf{x} : \mathbf{y} = 0\}$ does not contain complete trajectories of system (4.2) (the proof of this is similar to [3]). Consequently, if conditions (3.14) and (4.4) are satisfied, the unperturbed motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable as a whole [5].

Theorem 4. If the initial conditions $z_i(0)$ $(i = 1, 2, 3)$ of the linear system (3.11) are selected in accordance with (3.12) and (3.14), and condition (4.4) is satisfied, the solid body sunjected to moment (4.1), where $\boldsymbol{\mu}^\circ$ is determined by equalities (3.21) and (4.5) in the presence of gravitational forces either performs the motion

$$\boldsymbol{\omega} = \lambda\mathbf{H}_0, \qquad \mathbf{H}_0 = \mathbf{S}_0 \tag{4.6}$$

or asymptotically tends to such motion. Motion (4.6) is asymptotically stable according to Liapunov.

In concluding the author thanks V. V. Rumiantsev for stating the problem and constant interest in this work.

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ON A CLASS OF PERIODIC MOTIONS OF A SOLID BODY ABOUT A FIXED POINT

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Existence of a new set of periodic solutions of the problem of a heavy solid body motion about a fixed point is proved by the small parameter method of Poincaré. It is assumed that the body does not greatly differ from a body with a dynamic symmetry axis, and that the constant of integration of the moment of momentum is fairly small.

Let us consider the motion of a heavy solid body about a fixed point. The equation of motion of this problem can be reduced to a fourth order system describing the motion of a fictitious material point in a plane, by using the cyclic integral $\partial T / \partial\psi^{\cdot} = f$, where

T is the kinetic energy, ψ is the precession angle, and f is an arbitrary constant. To do this we effect two consecutive transformations of coordinates.

The first of these is defined by formulas

$$\varphi = \operatorname{Arctg} \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{sn} s}, \qquad \vartheta = \arccos \frac{\operatorname{dn} u \operatorname{cn} s}{\operatorname{dn} s}$$

where φ and ϑ are, respectively, the angles of spin and of nutation; u and s are the new variables with the elliptic functions of argument u having $k = [(A - B)C / (A - C)B]^{1/2}$, as their module, while the elliptic functions of argument s depend on module $k' = (1 - k^2)^{1/2}$.

The second transformation is defined by formulas (*)

$$x = \int_0^u \mu_1(u)\, du, \qquad y = -\int_0^s \mu_2(s)\, ds$$

$$\mu_1^2(u) = B + (A - B) \operatorname{sn}^2 u, \qquad \mu_2^2(s) = \frac{A - Bk'^2 \operatorname{sn}^2 s}{\operatorname{dn}^2 s}$$

The equations of motion are regularized by the introduction of the new variable defined by $dt = I d\tau$, where

$$I = \frac{Ck}{\Lambda}\left(k'^2 \frac{\operatorname{sn}^2 s}{\operatorname{dn}^2 s} + \operatorname{cn}^2 u\right)$$

$$\Lambda = \frac{1}{\operatorname{dn}^2 s}\left(Ak^2 \operatorname{sn}^2 u + Bk^2 \operatorname{cn}^2 u \operatorname{sn}^2 s + C \operatorname{dn}^2 u \operatorname{cn}^2 s\right)$$

As the result, we obtain the system of equations of motion

$$x'' - \frac{f\Omega}{k} y' = V_x', \qquad y'' + \frac{f\Omega}{k} x' = V_y' \tag{1}$$

where the prime denotes differentiation with respect to τ and the following notation is used:

$$V = \frac{I}{k}\left[h - \frac{mg}{\operatorname{dn} s}\left(ak \operatorname{sn} u + bk \operatorname{sn} s \operatorname{cn} u + c \operatorname{dn} u \operatorname{cn} s - \frac{f^2}{2\Lambda}\right)\right]$$

$$\Omega = \frac{1}{\mu_1(u)\mu_2(s)}\left\{\nu_2(s) \frac{\partial}{\partial u}\left[\frac{\lambda_1(u)}{\Lambda M}\right] + \nu_1(u) \frac{\partial}{\partial s}\left[\frac{\lambda_2(s)}{\Lambda M}\right]\right\}$$

$$\nu_1(u) = C \operatorname{dn}^2 u + k^2 (A - B) \operatorname{sn}^2 u \operatorname{cn}^2 u, \qquad \nu_2(s) = C \operatorname{cn}^2 s - k^2 (A - B) \frac{\operatorname{sn}^2 s}{\operatorname{dn}^2 s}$$

$$\lambda_1(u) = \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u, \qquad \lambda_2(s) = \frac{\operatorname{sn} s \operatorname{cn} s}{\operatorname{dn} s}, \qquad M = 1 - \operatorname{cn}^2 u \operatorname{cn}^2 s$$

where h is the constant of kinetic energy, and a, b and c are coordinates of the body center of gravity, attached to the latter.

System (1) has the Jacobi integral $x'^2 + y'^2 = 2V$ in which the arbitrary constant must be deleted.

The Poincaré method of small parameter can be used for proving the existence of periodic solution of Eqs. (1) and for their actual derivation. We restrict the analysis to the case of small f for which it is possible to set $f = k^3 f^*$, where f^* is the finite quantity. As the small parameter we take module k of elliptic functions, and construct the solution in the form of series

$$x(\tau) = x_0(\tau) + kx_1(\tau) + \ldots, \quad y(\tau) = y_0(\tau) + ky_1(\tau) + \ldots$$

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*) An error appears to have slipped in in the similar transformation in Arzhanykh's paper [1].

The simplified system of equations

$$x_0'' = \frac{\partial V_0}{\partial x_0}, \quad y_0'' = \frac{\partial V_0}{\partial y_0} \tag{2}$$

$$V_0 = (h - mgc)\left(\cos^2 \frac{x_0}{\sqrt{A}} + \operatorname{sh}^2 \frac{y_0}{\sqrt{A}}\right)$$

is obtained from (1) for $k = 0$. Its general solution can be written as

$$\cos \frac{x_0}{\sqrt{A}} = \operatorname{cn}(w_1, \varkappa), \quad \operatorname{sh} \frac{y_0}{\sqrt{A}} = \frac{\varkappa'}{\varkappa \operatorname{cn}(w_2, \varkappa)}$$

$$w_i = \sigma(\tau - \tau_i), \quad i = 1, 2; \ \sigma = \sqrt{\frac{C_1 + 2(h - mgc)}{A}},$$

$$\varkappa^2 = \frac{2(h - mgc)}{C_1 + 2(h - mgc)}$$

We consider here only the case of $h - mgc > 0$ and $C_1 > 0$.

The generating solution is obviously periodic of period $T = 4\sigma^{-1}K(\varkappa)$, where $K(\varkappa)$ is a complete elliptic integral of the first kind.

Let us pass to solving the equations in variations

$$x_1'' = \frac{\partial^2 V_0}{\partial x_0^2} x_1, \quad y_1'' = \frac{\partial^2 V_0}{\partial y_0^2} y_1 \tag{3}$$

Since system (2) is autonomous, system (3) admits the particular solution $x_1(\tau) = x_0'(\tau)$, $y_1(\tau) = y_0'(\tau)$. Hence by introducing variables $x_1 = x_0'\xi$, $y_1 = y_0'\eta$, we can write the first approximation equation in the form

$$x_0'^2\xi'' + (x_0'^2)'\xi' = f^*\Omega_0 x_0' y_0', \quad y_0'^2\eta'' + (y_0'^2)'\eta' = -f^*\Omega_0 x_0' y_0' \tag{4}$$

where Ω_0 is the value of function Ω for $k = 0$) .

From (4) we obtain the general solution of the first approximation equation

$$x_1 = x_0'\left\{\beta_1 \int \frac{d\tau}{x_0'^2} + \beta_2 + f^* \int \left(\int \Omega_0 x_0' y_0' \, d\tau\right) \frac{d\tau}{x_0'^2}\right\} \tag{5}$$

$$y_1 = y_0'\left\{\beta_3 \int \frac{d\tau}{y_0'^2} + \beta_4 - f^* \int \left(\int \Omega_0 x_0' y_0' \, d\tau\right) \frac{d\tau}{y_0'^2}\right\}$$

where β_1, β_2, β_3 and β_4 are constants of integration.

In accordance with the symmetry theorem in [2] the conditions of periodicity of solution of system (1) can be simplified. Since system (1) is invariant with respect to substitutions $\tau \to -\tau, \ x \to -x, \ y \to y, \ x' \to x', \ y' \to -y'$

the conditions of periodicity become

$$\psi_1 = x(0) = 0, \quad \psi_2 = x\left(\frac{T}{2}\right) = 0, \quad \psi_3 = y'(0) = 0, \quad \psi_4 = y'\left(\frac{T}{2}\right) = 0$$

They constitute a system of equations in parameters β_i which are compatible for

$$\left[\frac{D(\psi_1, \psi_2, \psi_3, \psi_4)}{D(\beta_1, \beta_2, \beta_3, \beta_4)}\right]_{k=0} \neq 0 \tag{6}$$

Taking into consideration the form of the first approximation equations (5) it is possible to transform condition (6) to the form

$$\left[\frac{D(\psi_1, \psi_2)}{D(\beta_1, \beta_2)}\right]_{k=0} \neq 0, \quad \left[\frac{D(\psi_3, \psi_4)}{D(\beta_3, \beta_4)}\right]_{k=0} \neq 0 \tag{7}$$

Carrying out integration in (5), we obtain

$$x_1 = \sigma \sqrt{A}\, \mathrm{dn}\, w_1 \left\{ \frac{1}{A\sigma^3 \varkappa'^2} \left[\mathrm{E}\,(\mathrm{am}\, w_1) - \frac{\varkappa^2 \,\mathrm{sn}\, w_1 \,\mathrm{cn}\, w_1}{\mathrm{dn}\, w_1} \right] \beta_1 + \beta_2 + \ldots \right\} \quad (8)$$

$$y_1 = \sigma \sqrt{A}\, \varkappa' \frac{\mathrm{sn}\, w_2}{\mathrm{cn}\, w_2} \left\{ -\frac{1}{A\sigma^3 \varkappa'^2} \left[\frac{\mathrm{cn}\, w_2 \,\mathrm{dn}\, w_2}{\mathrm{sn}\, w_2} + \mathrm{E}\,(\mathrm{am}\, w_2) \right] \beta_3 + \beta_4 + \ldots \right\}$$

Using (8) we reduce conditions (7) of existence of periodic solutions to the inequalities

$$\left[\frac{D(\psi_1, \psi_2)}{D(\beta_1, \beta_2)} \right]_{k=0} = -\frac{2}{\sigma \varkappa'^2} \mathrm{E}\,(\varkappa) \neq 0, \quad \left[\frac{D(\psi_3, \psi_4)}{D(\beta_3, \beta_4)} \right]_{k=0} = 2E\,(\varkappa)$$

which are satisfied for any periodic solutions escept for $\varkappa' = 0$.

Quasi-periodic motions of the solid body generally correspond to the derived periodic solutions of system (1).

In fact let us consider the cyclic integral

$$\psi' = \frac{If - C\varphi' \cos \vartheta}{(A \sin^2 \varphi + B \cos^2 \varphi) + C \cos^2 \vartheta}$$

By expanding its right-hand part in Fourier series in multiples of argument τ / T and integrating, we obtain $\psi = n(\tau - \tau_0) + \Phi(\tau)$, where $\Phi(\tau + T) = \Phi(\tau)$, and n is a constant quantity dependent on initial conditions. It is obvious that generally nT is not a multiple of 2π, which shows the validity of the above conclusion.

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ON THE STRUCTURE OF FORCES

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It is shown that a wide class of nonlinear forces can be represented by the sum of potential, nonconservative position, gyroscopic and dissipative forces.

Investigation of motion stability is in many instances conveniently achieved by analyzing the structure of acting forces. This method, whose basis was established in [1], has been recently successfully applied mainly to linear systems of arbitrary forces which can be fully represented as the sum of potential, nonconservative position, gyroscopic and dissipative forces. It is shown in this paper that such representation of forces can also be applied to a wide class of nonlinear forces.

Let an arbitrary vector field $\mathbf{Q}(x) = \mathbf{Q}(x_1, \ldots, x_n)$ be specified in some region of an n-dimensional orthogonal space $x = (x_1, \ldots, x_n)$. We call the vector field $\mathbf{R}(\mathbf{x})$ a circulation field, if at every point M of the specified field region vectors $\mathbf{R}$ and $\mathbf{x}$